Topological Method for the Colored Bárány's Theorem

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Let $P \subset \mathbb{R}^d$ be a set of n points. Every $d+1$ of them span a simplex, for a total of $\binom{n}{d+1}$ simplices. The point selection problem asks for a point contained in as many simplices as possible. Boros and Füredi showed for $d = 2$ that there always exists a point in $\mathbb{R}^2$ contained in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ simplices. A short and clever proof of this result was given by Bukh. Bárány generalized this result to higher dimensions:

Theorem (Bárány)**.** *There exists a point in $\mathbb{R}^d$ that is contained in at least $c_d\binom{n}{d+1} - O(n^d)$ simplices, where $c_d > 0$ is a constant depending only on the dimension d.*

This general result, the Bárány's theorem, is also known as the first selection lemma. We will henceforth denote by c_d the largest possible constant for which the Bárány's theorem holds true. Bukh, Matoušek and Nivasch used a specific construction called the stretched grid to prove that the constant $c_2 = \frac{2}{9}$ in the planar case found by Boros and Füredi is the best possible. In fact, they proved that $c_d \leqslant \frac{d!}{(d+1)^d}$. On the other hand, Bárány's proof in implies that $c_d \geqslant (d+1)^{-d}$, and Wagner improved it to $c_d \geqslant \frac{d^2+1}{(d+1)^{d+1}}$.

Gromov further improved the lower bound on c_d by topological means. His method gives $c_d \geqslant \frac{2d}{(d+1)(d+1)!}$. Karasev found a very elegant proof of Gromov's bound, which he described as a "decoded and refined" version of Gromov's proof.

In this talk, we are concerned with a colored variant of the point selection problem. Let $P_0, \ldots, P_d$ be $d+1$ disjoint finite sets in $\mathbb{R}^d$. A *colorful simplex* is the convex hull of $d+1$ points each of which comes from a distinct P_i. For the colored point selection problem, we are concerned with the point(s) contained in many colorful simplices. We discuss Karasev's topological proof for the colored variant and our slight improvement on the bound.