SMOOTHNESS FOR HIGH-ORDER CONNECTIVITY IN RANDOM HYPERGRAPHS

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Given integers $1 \leq j < k$ and a k-uniform hypergraph $\mathcal{H} = (V, E)$ (where $E \subseteq \binom{V}{k}$), we define the notion of *j-connectivity* as follows: Two j-sets J, J' are j-connected if we can walk from J to J' using edges that consecutively intersect in at least j vertices. A j-component is a maximal set of pairwise j-connected j-sets. The case $j = 1$ and $k = 2$ corresponds to connectedness in graphs. More generally, the case $j = 1$ is known as vertex-connectivity.

The study of high-order connectivity (i.e. $j > 1$) is motivated by related, though not equivalent notions of the vanishing of the $(j-1)$-th homology group in simplicial complexes, or loose and tight cycles in hypergraphs. These cycles roughly correspond to the cases $j = 1$ and $j = k - 1$, and indeed j-tight cycles have been studied for each $1 \leq j \leq k - 1$.

In terms of connectivity in random hypergraphs, however, the case $j = 1$ has received by far the most attention despite the fact that larger j exhibit far richer, more complex phenomena.

The difficulty with larger j - beyond the problems of visualisation - is often that the j-sets of a structure may be "badly distributed" in that they are highly correlated in an uncontrollable way, rather than spread evenly over the hypergraph.

We present a tool, the *smoothness lemma*, which first appeared in [1] and which guarantees that in the binomial random hypergraph, with high probability various collections of j-sets (e.g. large components, unions of small components, boundaries of component search processes etc.) are evenly distributed in the following sense: For any $1 \leq \ell \leq j - 1$ and for any ℓ-set L, the number of j-sets containing L is about the "right" number. The version we present here is significantly more general and user-friendly than the version in [1]. We argue that it can be used as a black box to be applied to many related problems in hypergraph theory.

We mention two results to which it has already been applied:

- The original application was to determine the asymptotic size of the giant j-component close to the phase transition threshold [1].
- A second application was to determine a threshold and hitting time result for the hypergraph to be completely j-connected. [2]

These results and their proofs will be expanded upon in a second talk by Christoph Koch. This talk is based on joint work with Mihyun Kang and Christoph Koch.

References

1. O. Cooley, M. Kang, and C. Koch, *The size of the giant component in random hypergraphs*, Submitted. Arxiv number 1501.07835.
2. ———, *Threshold and hitting time for high-order connectivity in random hypergraphs*, Submitted. Arxiv number 1502.07289.