RANDOM GRAPH COLORINGS

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ABSTRACT. Going back to the seminal paper of Erdős and Rényi that founded the theory of random graphs, the problem of coloring $\boldsymbol{G} = \boldsymbol{G}(n, m)$ remains one of the longest-standing challenges in probabilistic combinatorics. The graph coloring problem is profoundly rule-governed by the interplay between local and far reaching global effects. Given a k-coloring $\boldsymbol{\sigma}$ of $\boldsymbol{G}$, the colors that $\boldsymbol{\sigma}$ assigns to the neighbors of a vertex v and the color of v are correlated in that they must be distinct. It seems reasonable to expect that for any fixed "radius" ω the colors assigned to the vertices at distance ω from v and the color of v itself will typically be correlated. Keeping in mind that locally around almost any vertex the sparse random graph is bipartite w.h.p., while globally the chromatic number of the random graph may be large, we address the question whether these correlations persist as ω goes to infinity. As pointed out by Erdős, the random graph $\boldsymbol{G}$ provides therefor an example of a graph that simultaneously has a high chromatic number and a high girth.
According to a prediction from statistical physics, for average degrees below the so-called *condensation threshold* $d_{k,\mathrm{cond}}$, the colors assigned to far away vertices are asymptotically independent [Krzakala et al.: Proc. National Academy of Sciences 2007]. We prove this conjecture for k exceeding a certain constant k_0 by introducing the concept of *planting replicas* as an extension of the "planting trick" that was introduced by Achlioptas and Coja-Oghlan [Proc. 49th FOCS 2008].

To state our main result, for a k-colorable graph $G = (V, E)$, a vertex v and an integer $\omega \geq 0$ we let $\partial^{\omega}(G, v)$ signify the depth-ω neighborhood of v, i.e., the graph obtained from G by deleting all vertices at a distance greater than ω from v. Additionally, for a set $U \subset V$ we let $\mu_{k,G|U}$ denote the projection of $\mu_{k,G}$, the uniform distribution on the set of k-colorings of G, onto $[k]^U$, i.e.,

$$\mu_{k,G|U}(\sigma_0) = \mu_{k,G}\left(\left\{\sigma \in [k]^V : \forall u \in U : \sigma(u) = \sigma_0(u)\right\}\right) \qquad (\sigma_0 \in [k]^U).$$

Additionally, given a graph G, we let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ denote vertices of G that are chosen uniformly and independently at random. Finally, let $\|\cdot\|_{\mathrm{TV}}$ be the total variation norm.

Theorem. *There is a constant $k_0 > 0$ such that for any $k \geq k_0$, $d < d_{k,\mathrm{cond}}$, $l \geq 1$, $\omega \geq 0$ we have*

$$\lim_{n\to\infty} \mathbb{E}\left\| \mu_{k,\boldsymbol{G}|V(\partial^{\omega}(\boldsymbol{G},\boldsymbol{v}_1)\cup\cdots\cup\partial^{\omega}(\boldsymbol{G},\boldsymbol{v}_l))} - \bigotimes_{i=1}^{l} \mu_{k,V(\partial^{\omega}(\boldsymbol{G},\boldsymbol{v}_i))} \right\|_{\mathrm{TV}} = 0.$$

In addition, we point out an implication on the *reconstruction problem.* In general, the reconstruction problem considers the bias on the assignments of vertices on a *fixed* "radius" ω from v. If this correlation persists as $\omega \to \infty$ we say that *reconstruction occurs* on $\boldsymbol{G}$. A similar notion of *tree reconstruction* can be defined on a random Galton-Watson tree $\boldsymbol{T}(k, d)$ with offspring distribution $\mathrm{Po}(d)$.

Theorem. *There is a number $k_0 > 0$ such that for all $k \geq k_0$ and $d < d_{k,\mathrm{cond}}$ the following is true.*

Reconstruction is possible on $\boldsymbol{G}$ $\Leftrightarrow$ tree reconstruction is possible on $T(k, d)$.

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