Rank deficiency in sparse random GF[2] matrices

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Abstract

Let M be a random $m \times n$ matrix with binary entries and i.i.d. rows. The weight (i.e., number of ones) of a row has a specified probability distribution, with the row chosen uniformly at random given its weight. Let $\mathcal{N}(n,m)$ denote the number of left null vectors in $\{0,1\}^m$ for M (including the zero vector), where addition is mod 2. We take $n, m \to \infty$, with $m/n \to \alpha > 0$, while the weight distribution converges weakly to that of a random variable W on $\{3,4,5,\ldots\}$. Identifying M with a hypergraph on n vertices, we define the *2-core* of M as the terminal state of an iterative algorithm that deletes every row incident to a column of degree 1.

We identify two thresholds α^* and $\underline{\alpha}$, and describe them analytically in terms of the distribution of W. Threshold α^* marks the infimum of values of α at which $n^{-1} \log \mathbb{E}[\mathcal{N}(n,m)]$ converges to a positive limit, while $\underline{\alpha}$ marks the infimum of values of α at which there is a 2-core of non-negligible size compared to n having more rows than non-empty columns.

We have $1/2 \leq \alpha^* \leq \underline{\alpha} \leq 1$, and typically these inequalities are strict; for example when $W = 3$ almost surely, $\alpha^* \approx 0.8895$ and $\underline{\alpha} \approx 0.9179$. The threshold of values of α for which $\mathcal{N}(n,m) \geq 2$ in probability lies in $[\alpha^*, \underline{\alpha}]$ and is conjectured to equal $\underline{\alpha}$. The random row-weight setting gives rise to interesting new phenomena not present in the case of non-random W that has been the focus of previous work.

Key words: Random sparse matrix, null vector, hypercycle, random allocation, XORSAT, phase transition, hypergraph core, large deviations, Ehrenfest model

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References

[1] R W R Darling, Mathew Penrose, Andrew Wade, Sandy Zabell Rank deficiency in sparse random GF[2] matrices, Electronic Journal of Probability 19, Art. 83, 2014

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