Lubell mass and induced partially ordered sets

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Abstract

Let P be a partially ordered set. Let Q_n denote the hypercube of dimension n ordered by inclusion. A subset $\mathcal{F}$ of Q_n is said to contain P *weakly* if there exists an injective map $\psi : P \longrightarrow \mathcal{F}$ such that, if $x, y \in P$, then $x \leq y$ implies $\psi(x) \subseteq \psi(y)$. $\mathcal{F}$ contains P *strongly* if there exists an injective map $\psi : P \longrightarrow \mathcal{F}$ such that, if $x, y \in P$, then $x \leq y$ if and only if $\psi(x) \subseteq \psi(y)$.

In this talk we consider a Turán-type question for partially ordered sets: given a fixed poset P, what is the maximal size of a subset of Q_n which does not contain P? Erdős extended Sperner's Lemma by proving that if $\mathcal{F}$ does not contain a chain of length k then $|\mathcal{F}|$ is at most the sum of the $k-1$ largest binomial coefficients of order n. It easily follows that $|\mathcal{F}| \leq (|P|-1)\binom{n}{\lfloor n/2 \rfloor}$ if $\mathcal{F}$ does not weakly contain P. For strong containment the situation is more complicated and it is only recently that Methuku and Pálvölgyi proved that if $\mathcal{F}$ does not contain P strongly then $|\mathcal{F}| \leq c(P)\binom{n}{\lfloor n/2 \rfloor}$ for some constant $c(P)$. Their proof relies on a generalization of the Marcus-Tardos theorem about forbidden permutation matrices in 0-1 matrices.

A generalization of the LYM inequality is the fact that if $\mathcal{F}$ does not contain a chain of length k then $\sum_{F \in \mathcal{F}} 1/\binom{n}{|F|} \leq k-1$, from which it again follows that $\sum_{F \in \mathcal{F}} 1/\binom{n}{|F|} \leq |P|-1$ if $\mathcal{F}$ does not weakly contain a fixed poset P. We prove a corresponding result for strong containment: there exists a constant $c(P)$ such that if $\mathcal{F}$ does not strongly contain P then $\sum_{F \in \mathcal{F}} 1/\binom{n}{|F|} \leq c(P)$. This strengthens Methuku and Pálvölgyi's result and the argument is very different — it relies on some density arguments combined with concentration inequalities for random variables.

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