Universal Zero-One k–Law

A.D. Matushkin

The random graph $G(n,p)$ *obeys Zero-One Law* if for each first-order property its probability tends to 0 or tends to 1. In [1] S. Shelah and J. Spencer showed that if α is an *irrational* positive number and $p = n^{-\alpha+o(1)}$, then $G(n,p)$ obeys Zero-One Law. In the same paper it was proved that if $\alpha \in (0,1]$ is a *rational* number, then $G(n,n^{-\alpha})$ does not obey Zero-One Law.

We let $\mathcal{L}_k$ and $\mathcal{L}$ denote the set of properties which are expressed by first-order formulae with quantifier depth at most k and the set of all first-order properties respectively. We say that the random graph $G(n,p)$ *obeys Zero-One k–Law* if for each $L \in \mathcal{L}_k$ its probability tends to 0 or tends to 1.

In [2, 3] it was shown that the minimal and the maximal numbers α such that random graph $G(n,n^{-\alpha})$ *does not* obey Zero-One k–Law are $\frac{1}{k-2}$ and $1-\frac{1}{2^k-2}$ respectively.

Consider rational points $\alpha \in (0,1)$ such that Zero-One k–Law does not hold in their neighborhood. We say that $\alpha \in \mathbb{Q}$ is a *k–critical point* if for some $A \in \mathcal{L}_k$ the following property is *not* satisfied. There exists $\delta \in \{0,1\}$ and $\varepsilon > 0$ so that $\lim_{n\to\infty} \mathsf{P}_{n,p}(A) = \delta$ holds for all $p \in [n^{-\alpha-\varepsilon}, n^{-\alpha+\varepsilon}]$. It is known [1] that the set of k–critical points is well ordered under the order $>$. Hence, there is an interval without any k–critical points to the left of any rational point (in particular, the Zero-One k–Law holds on this interval). We found such an interval explicitly for any $k \geq 4$ and any rational number $t/s \in (0,1)$.

Theorem 1. *Let* $\frac{t}{s} \in \mathbb{Q} \cap (0,1)$, $k \geq 4$. *Denote* $q = \frac{(s+1)^k-1}{s}$. *Then there are no k–critical points inside the interval* $\left(\frac{tq}{sq+1}, \frac{t}{s}\right)$.

We also proved, that if t/s is a rational number with numerator not greater than 2, then logarithm of our interval's length has the same asymptotics up to a constant factor (when $n \to \infty$) as logarithm of the biggest interval with right end at (t/s) without k–critical points.

Theorem 2. *Let* $\frac{t}{s} = \frac{2}{m}$, $m \geq 2$, $k \geq 10m-5$. *Then for* $\alpha = \frac{t}{s} - \frac{1}{2^{k-10m+8}m(m-1)}$ *the random graph* $G(n,n^{-\alpha})$ *does not obey Zero-One k–Law.*

References

[1] S. Shelah, J.H. Spencer, *Zero-one laws for sparse random graphs*, J. Amer. Math. Soc., **1**: 97–115, 1988.

[2] M.E. Zhukovskii, *Zero-one k-law*, Discrete Mathematics, **312**: 1670–1688, 2012.

[3] M.E. Zhukovskii *The largest critical point in the zero-one k–law*, Sbornik: Mathematics (2015), 206(4): 489.