On infinite spectra of first-order properties

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The random graph $G(n,p)$ *obeys Zero-One Law* if for each first-order property its probability tends to 0 or tends to 1. In [1], S. Shelah and J. Spencer showed that if α is an *irrational* positive number and $p(n)=n^{-\alpha+o(1)}$, then $G(n,n^{-\alpha})$ obeys Zero-One Law.

We let $\mathcal{L}_k$ and $\mathcal{L}$ denote the set of properties which are expressed by first-order formulae with quantifier depthes at most k and the set of all first-order properties respectively. For any $L\in\mathcal{L}$, we define two notions of its spectra, $S^1(L)$ and $S^2(L)$. The first considers $p=n^{-\alpha}$. $S^1(L)$ is the set of $\alpha\in(0,1)$ which does *not* satisfy the following property: $\lim_{n\to\infty}\mathsf{P}(G(n,n^{-\alpha})\models L)$ exists and is either zero or one. The second considers $p=n^{-\alpha+o(1)}$. $S^2(L)$ is the set of $\alpha\in(0,1)$ which does *not* satisfy the following property: there exists $\delta\in\{0,1\}$ and $\epsilon>0$ so that when $n^{-\alpha-\epsilon}<p(n)<n^{-\alpha+\epsilon}$, $\lim_{n\to\infty}\mathsf{P}(G(n,p(n))\models L)=\delta$. It can be shown that for any rational $\alpha\in(0,1)$ there is a first-order property L such that $\alpha\in S^1(L)$. So, the Zero-One Law of Shelah and Spencer implies $\bigcup_{L\in\mathcal{L}}S^1(L)=\mathbb{Q}\cap(0,1)$. It is easy to show that letting L be the property of every two vertices to have a common neighbor, $S^2(L)=\{\frac{1}{2}\}$ while $S^1(L)=\varnothing$. Thus, for any $L\in\mathcal{L}$, we have $S^1(L)\subset S^2(L)$ but there may not be the equality. However, in [1] Shelah and Spencer proved that every $S^2(L)$ consists only of rational numbers as well.

Let S^1_k be the union of all $S^1(L)$ where $L\in\mathcal{L}_k$, S^2_k be the union of all $S^2(L)$ where $L\in\mathcal{L}_k$. In [2, 3], it was shown that the minimal and the maximal numbers in S^1_k equal $\frac{1}{k-2}$ and $1-\frac{1}{2^k-2}$ respectively. In [4], it was proved that the sets S^1_k and S^2_k are infinite when k is large enough. It is also known [5] that all limit points of S^1_k and S^2_k are approached only from above. For any $j\in\{1,2\}$, denote the set of limit points of S^j_k by $(S^j_k)'$. In our joint work with Spencer, we prove that if $k\geq 15$, then $\min(S^1_k)'\leq\frac{1}{k-11}$. If $k\geq 10$, then $\min(S^2_k)'\leq\frac{1}{k-7}$. If $k\geq 16$, then $\max(S^2_k)'\geq\max(S^1_k)'\geq 1-\frac{1}{2^{k-13}}$. Moreover, we prove that the minimal k_1 and k_2 such that $S^1_{k_1}$ and $S^2_{k_2}$ are infinite are from $\{4,\ldots,12\}$ and $\{4,\ldots,10\}$ respectively. Recently, we significantly improve this result.

Theorem 1 *If* $k\geq 5$, *then* $\frac{1}{[k/2]}\in(S^1_k)'$. *If* $k\geq 8$, *then* $\max(S^2_k)'\geq\max(S^1_k)'\geq 1-\frac{1}{2^{k-5}}$.

Consequently, the minimal k such that the set S^1_k (S^2_k) is infinite is either 4 or 5.

References

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