Random Trees obtained from Permutation Graphs

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Abstract

A permutation graph is an undirected graph obtained from a permutation by drawing an edge for each inversion in the permutation. More formally, given a permutation $\boldsymbol{w} = w_1, \ldots, w_n$ of $[n] := \{1, \ldots, n\}$, the permutation graph $G_{\boldsymbol{w}}$ is defined to be the (undirected) graph with the vertex set $[n]$ and the edge set $\{(w_a, w_b) : (w_a, w_b) \text{ is an inversion}\}$ (a pair (w_a, w_b) is an *inversion*, if $a < b$ and $w_a > w_b$).

This definition was given by Even, Pnuelli, and Lempel in 1972 and it differs from the one given by Chartrand and Harary in 1967.

Since a permutation is uniquely determined by the set of its inversions, two different permutations yield two different graphs and thus there are $n!$ permutation graphs on the vertex set $[n]$, as opposed to $2^{\binom{n}{2}}$ general graphs.

Permutation graphs form a subclass of perfect graphs and consequently various NP-complete problems in general graphs, including the coloring problem, the maximum clique problem, and the maximum independent set problem, have polynomial time solutions in permutation graphs. This aspect of permutation graphs has led to many studies that are computational in nature. Frequently, a problem on permutation graphs can be easily translated to a problem on permutations. For example, a clique in a permutation graph corresponds to a decreasing subsequence in the accompanying permutation and an independent set corresponds to an increasing subsequence.

Likewise, as was shown by Koh and Ree in 2007, connected permutation graphs correspond to indecomposable permutations (first studied by Lentin (1972) and Comtet (1972)). Recall that a permutation $\boldsymbol{w} = w_1, \ldots, w_n$ of $[n]$ is called *decomposable* at m if $\{w_1, \ldots, w_m\} = \{1, \ldots, m\}$ for $m < n$. If there is no such m, then $\boldsymbol{w}$ is called *indecomposable*.

In this talk, we present some recent results on permutation trees (i.e. permutation graphs that happen to be trees). We first find that the number of trees among permutation graphs on n vertices is 2^{n-2} (by Cayley formula, the number of all trees on n vertices is n^{n-2}). Then we study the asymptotic properties of a random permutation tree, i.e. a tree T_n that is chosen uniformly at random from all 2^{n-2} permutation trees as n tends to infinity. In particular, we study the degree distribution of T_n, the maximum degree in T_n, the diameter of T_n, and the size of a minimum dominating set in T_n. We find that the number of leaves and the diameter are binomially distributed. We also find the asymptotic distribution of the maximum degree in T_n as $n \to \infty$. Further, denoting by D_i the number of degree-i vertices in T_n, we prove that $(D_i)_{i=1}^{m}$ is asymptotically jointly normal for any m. Finally, we show that the size of a minimum dominating set, $\gamma(T_n)$, is also asymptotically normally distributed with mean $n/3 + O(1)$ and variance $0.26n + O(1)$.

The talk is based on a joint work with Hüseyin Acan.